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Comparison of solutions for cavity expansion in cohesive frictional soils

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ABSTRACT

This paper presents a comparison of the solutions for cavity expansion in cohesive frictional soils, which are modelled as elastic perfectly plastic materials obeying the Mohr Coulomb failure criterion. Vesic's solutions were extended to the cavity expansions from a finite radius. A numerical procedure was proposed to compute the pressure expansion curve for cavity expansions using Hill's incremental velocity approach. It is shown that the numerical results obtained from the proposed procedure are equivalent to the analytical solutions by Yu and Houlsby. The analytical solutions are also validated against finite difference analysis using the explicit Lagrangian calculation scheme.

KEY WORDS: Cavity expansion; plasticity; cohesive frictional soils; large strains.

INTRODUCTION

Cavity expansion solutions have been widely used in the analysis of various practical problems in geotechnical engineering. Typical examples include the estimation of bearing capacity of piles, interpretation of pressuremeter and cone penetration tests, and ground response due to tunneling. A large number of solutions have been reviewed and given by Yu (2000). Among them, Yu and Houlsby (1991) developed a unified analytical solution for both cylindrical and spherical cavities expansion from a finite radius in elastic-plastic soils obeying the Mohr-Coulomb yield criterion with a non-associated flow rule. In their study, the total strain method proposed by Chadwick (1959) was followed and the finite logarithmic strain was adopted in the plastic zone. Later, Yu and Carter (2002) derived a rigorous closed-form similarity solution for the expansion of cavities in cohesive-frictional soils, satisfying the Mohr-Coulomb yield criterion, in which the incremental velocity method first adopted by Hill (1950) was used and the plastic radius was taken as the "time" scale for the progress of the cavity. The similarity solution, i.e., the limit solution for cavity expansion from zero radius, can be taken as the intermediate asymptote if the cavity expands from a finite radius to infinity. They termed the two methods as the Lagrangian and Eulerian method, respectively. They also demonstrated that the numerical results in terms of limiting pressure and plastic radius calculated using the two methods are practically identical. In addition, Vesic (1972) proposed a solution for the general cases of cylindrical and spherical cavities in an infinite Mohr Coulomb material. The volume change in the plastic zone was accounted for by using an average volumetric strain. Because the initial cavity radius was assumed as zero, Vesic's solution can only be taken as a similarity solution and the associated limit pressure significantly relies on the accurate estimation of the average volumetric strain. These solutions have been used as benchmarks for numerical analysis involving material and geometric nonlinearities and large strains (Carter and Yeung, 1985; Shuttle and Jefferies, 2016; Guo et al., 2016).

This paper aims to improve the understanding of the solutions for cavity expansion in cohesive frictional soils. The specific objectives of this paper are to: (1) extend the Vesic solution for a cavity expanding from a finite radius; (2) propose a numerical procedure for computing the pressure expansion curve for cavity expansion from a finite radius using Hill's incremental velocity approach; and (3) compare the pressure expansion curves obtained from different analytical approaches as well as numerical analysis.

The following assumptions have been made with respect to the geometry, initial and boundary conditions, and constitutive relations: (1) A cylindrical (k=1) or spherical (k = 2) cavity of initial radius a_0 is embedded in an infinite, homogeneous, isotropic and weightless soil. (2) The initial stress field is a uniform hydrostatic stress state of magnitude p_0. (3) The cavity is expanded from initial radius a_0 to a current radius a with the internal pressure increasing from p_0 to p, while the stresses in the far field remain unaltered by the cavity expansion. (4) The analysis is for cohesive frictional soils under dry or fully drained conditions. (5) The soil is modelled as an elastic perfectly plastic material obeying the Mohr Coulomb failure criterion, with the soil properties defined by Young's modulus E and Poisson's ratio v, and the cohesion c, and angles of friction and dilation ϕ and ψ. (6) A large-strain analysis in the plastic region and a small-strain solution in the elastic region are adopted.

The problem involves both geometric and material nonlinearities and has been studied by Vesic (1972), Carter et al. (1986), Yu and Houlsby (1991) and Yu and Carter (2002), among others. The details such as stress analysis and limit pressure will not be repeated here. This paper focuses on the pressure expansion relationship. Hence, only the necessary results are reproduced so that the reader may follow the analysis in the next sections more readily. Upon increasing the cavity pressure from p_0, the soil around the cavity remains purely elastic until initial yield starts at the cavity wall. At this "time", the cavity radius $a=a_1$, and pressure $p=p_1$. With further increasing of the cavity pressure, a plastic zone within the region $a \le r \le R$ forms around the cavity wall. The soil remains elastic beyond the elasto-plastic interface, i.e r > R. The solutions at the elasto-plastic interface r=R serve as the continuity condition. The relevant mathematical functions and results are presented in the appendix.

GENERALISED VESIC SOLUTION

By following Vesic (1972), during the elastic-plastic stage of expansion when $p > p_1$, the kinematics of the cavity expansions requires

$$a^{k+1} - a_0^{k+1} = R^{k+1} - (R - u_R)^{k+1} + (R^{k+1} - a^{k+1})\Delta \tag{1}$$

Neglecting the higher powers of u_R, Eq. (1) becomes

$$\left(\frac{R}{a}\right)^{k+1} = \frac{1+\Delta}{(k+1)\varepsilon_R + \Delta}\left[1 - \frac{1}{1+\Delta}\left(\frac{a_0}{a}\right)^{k+1}\right] \tag{2}$$

where u_R and ε_R are the radial displacement and circumferential strain at the elastic-plastic boundary (r=R), Δ is the average volumetric strain in the plastic region. It is noted that ε_R doesn't vary with the expansion of the cavity and is defined by

$$\varepsilon_R = \frac{u_R}{R} = \frac{p_1 - p_0}{2kG} \tag{3}$$

The cavity pressure can be calculated from R/a in Eq (2) using the equation derived by Carter et al. (1986)

$$\frac{p + c\cot\phi}{p_0 + c\cot\phi} = \frac{1+k}{\alpha + k}\alpha\left(\frac{R}{a}\right)^{k\frac{\alpha-1}{\alpha}} \tag{4}$$

where $\alpha = (1+\sin\phi)/(1-\sin\phi)$. In combination with the elastic expansion when the cavity pressure $p < p_1$, Eqs (2) and (4) can be used to obtain the complete pressure expansion curve. If taking k=1, c=0 and $\Delta = 0$, it means a cylindrical cavity expansion in a purely frictional incompressible soil. The pressure expansion relationship will become that obtained by Gibson and Anderson (1961) for sands. If the cavity expands to infinity or from zero radius, i.e. $a_0/a=0$, and putting $\sqrt[(k+1)]{1+\Delta} \approx 1$ and $(3-\sin\phi)/3\cos\phi \approx 1$ for $\Delta < 0.15$ and $0 < \phi < 45°$ as proposed by Vesic (1972), the limit plastic radius can be obtained from Eqs. (2) and (3) as $R/a = \sqrt[3]{I_{rr}}$ and $R/a = \sqrt{I_{rr}^{'}\sec\phi}$ for spherical and cylindrical cavity, respectively, in which I_{rr} and $I_{rr}^{'}$ are the corresponding reduced rigidity index. These are the results originally presented by Vesic (1972).

The average volumetric strain, Δ, which characterizes the soil compressibility due to compression and distortion, plays an important role in the estimation of the expansion pressure relationship and the limit cavity pressure. Generally, it can be determined based on the results of isotropic compression tests and standard triaxial compression tests. However, it is convenient for practical applications to have analytical procedure to evaluate the average volumetric strain. Two such procedures have been proposed. Mayne and Kulhawy (1990) show within $25° < \phi < 45°$, the average volumetric strain, Δ, may be estimated from

$$\Delta = 0.005\left(1 - \frac{\phi - 25}{20}\right)\frac{\sigma_v'}{p_a} \tag{5}$$

in which σ_v' is the vertical effective stress, p_a is the atmospheric pressure. Yasufuku et al. (2001) found that the average volumetric strain, Δ, may be approximately determined from the rigidity index, I_r for sands, ranging from relatively incompressible siliceous sands to compressible carbonate sands, using an empirical equation

$$\Delta = 50\left(I_r\right)^{-1.8} \tag{6}$$

Eq (6) implies that the average volumetric strain reduces drastically with the increasing of rigidity index. At I_r =400, $\Delta \approx 0$, implying that the soil is almost incompressible.

CAVITY EXPANSION FROM A FINITE RADIUS

Following Hill's incremental velocity approach, Yu and Carter (2002) derived a complete analytical solution for the expansion of cavities from zero initial radius in an infinite cohesive-friction soil mass. The same approach was also adopted by Carter et al. (1986) in their study, but the convected part of the stress rate in the governing equation was neglected, hence resulting in only approximate solutions. Some salient characteristics of the problem when the cavity is expanded from zero radius, which is also referred to as the created cavity problem, are that it has no characteristic length and the cavity expands in a geometrically self-similar manner at a constant cavity pressure (Hill, 1950; Collins and Stimpson, 1994). Carter et al. (1986) stated that the entire pressure expansion curve can be obtained for their solution using a numerical technique. Nevertheless, it is not shown in Carter et al. (1986). Huang and Detournay (2010) derived a solution for the pressure expansion relationship of a cylindrical cavity expansion from a finite radius using the rate formulation, in which the current cavity radius is chosen as the "time" scale. But their solution requires solving a first ordinary differential equation. It is shown in this section that the pressure-expansion relationship for cavity expansions from a finite radius can be obtained directly using a numerical procedure from the equations presented by Yu and Carter (2002).

Taking the movement of the plastic boundary as the scale of "time" or progress of the expansion, Yu and Carter (2002) show that when the plastic boundary moves outward a distance dR, the cavity wall displacement, da, can be calculated by

$$da = V_{r=a}dR \tag{7}$$

The expression for the velocity at cavity wall $V_{r=a}$ is provided in the appendix. It depends on the properties of the soils, the initial in situ stress, and the ratio of plastic radius to the current cavity radius, R/a. Particularly, the velocity at the plastic boundary $V_{r=R}$ is known from the solution for the displacement in the elastic region and can be expressed as

$$V_{r=R} = \varepsilon_R\left(1+k\right) \tag{8}$$

It should be noted that Eq. (8) also applies at the onset of yielding starting at the cavity wall (i.e. when $p = p_1$ and $R=a_1$) and serves as the initial condition for expansion of the cavity afterwards. For a given small distance increment, dR, an initial displacement increment, da, is computed from $da = \varepsilon_R(1+k)dR$ at the first step. The dR and da are added to the R and a (both equal to a_1 at this step) to give their new values. They are used to calculate the cavity pressure from Eq. (4) and the velocity at cavity wall $V_{r=a}$ (Eq. (9) in the appendix). Then the plastic radius, R, and cavity radius, a, are updated by adding to the old values, increments dR and da calculated from Eq. (7). The procedure is repeated until the cavity pressure approaches asymptotically to the limit cavity pressure at large displacement. Together with the elastic expansion when the cavity pressure p is less than the pressure p_1, the entire pressure expansion curve can be plotted.

RESULTS AND DISCUSSIONS

To validate the analytical solutions, numerical analysis using FLAC (Itasca Consulting Group, Inc.), a two-dimensional explicit finite difference program, has been conducted. FLAC uses the explicit Lagrangian calculation scheme. A plane-strain model with the plane of analysis oriented normal to the axis of the cavity is created for the cylindrical cavity. The same model grid but with axisymmetric configuration is used for the spherical cavity model. Because of symmetry, only a quarter of the problem needs to be analyzed as shown in Figure 1. The left and the bottom boundaries are subjected to roller boundary conditions. Initially, in-situ stresses are installed. The cavity boundary is fixed and a pressure boundary condition of magnitude p_0 is applied at the outer radial boundary. A compressive velocity of magnitude $10^{-5}a_0$ is applied normally at the cavity boundary for a total of 10^6 steps to expand the cavity up to a/a_0=11. To simulate large deformation, the large strain mode is specified. To model a cavity expansion in an infinite soil, the outer boundary is set to a very large distance, generally greater than $500a_0$.

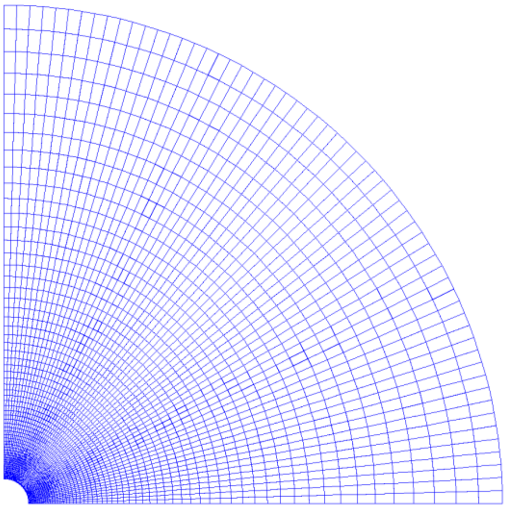

Figure 1   FLAC grid geometry for the model

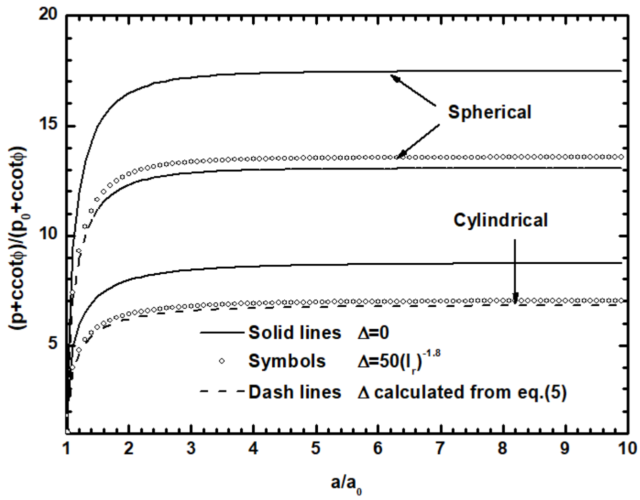

Figure 2 Pressure expansion curves for Vesic solutions

The soil parameters are taken from Yu and Houlsby (1991): Poisson's ratio v=0.3, frictional angle $\phi = 30°$, $\psi = 0°, 15°, \text{and } 30°$, $E/(p_0 + c\cot\phi) = 260$. The cavity pressure and expansion are expressed in dimensionless terms as $(p + c\cot\phi)/(p_0 + c\cot\phi)$ and a/a_0, respectively. Figure 2 shows the pressure-expansion curves for the generailised Vesic solution using the average volumetric strain calculated from Eq. (5) and (6) as well as Δ=0. At large strains, the cavity pressure approaches asymptotically to the limit pressure. It also demonstrates the significant influence of the average volumetric strain, indicating that one cannot simply assume Δ=0 in estimating the pressure-expansion curve and the limiting pressure. Figures 3~4 plot the pressure-expansion curves of cylindrical and spherical cavities from the proposed numerical procedures. The results from the solution by Yu and Houlsby (1991) are also plotted for comparison. The two solutions agree well with each other and the curves are indistinguishable. The calculated pressure-expansion curves from FLAC closely agree with those computed from the analytical solutions. The difference may be due to the highly non-uniform displacement field developed, the boundary condition, formation of shear bands and strain localization and other factors in the numerical model as shown by Guo et al. (2016). Figure 5 shows the variations of the velocity at cavity wall $V_{r=a}$ and the ratio a/R of the current cavity radius to the plastic radius with expansion ratio a/a_0 for a cylindrical cavity with $\phi = 30°, \psi = 10°, G/p_0 = 1000$. The numerical results confirm that at the large displacements a pseudo steady state configuration will be approached and ultimately have $V_{r=a} = da/dR = a/R$ as established in the similarity solutions by Carter et al. (1986) and Yu and Carter (2002).

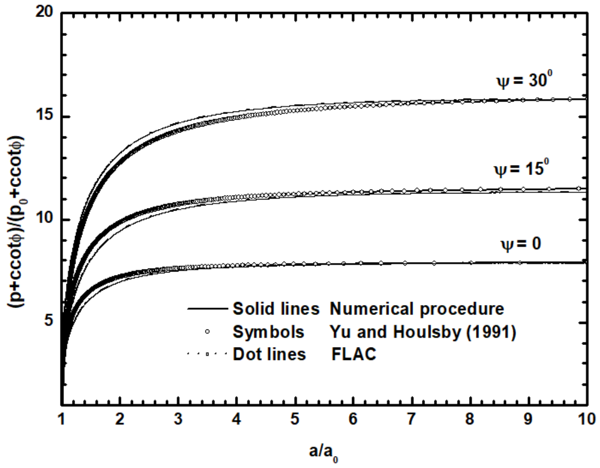

Figure 3 Pressure expansion curves for cylindrical cavities

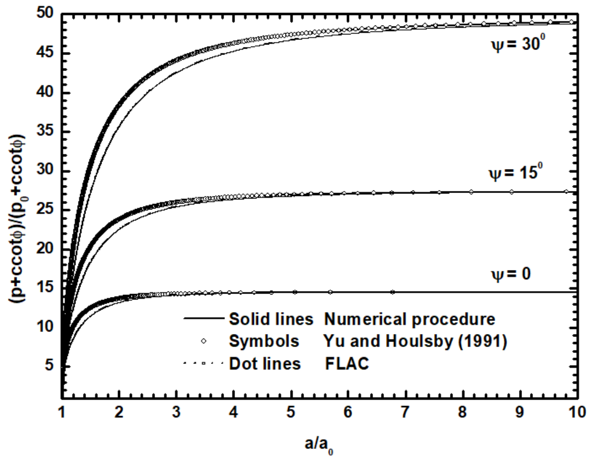

Figure 4 Pressure expansion curves for spherical cavities

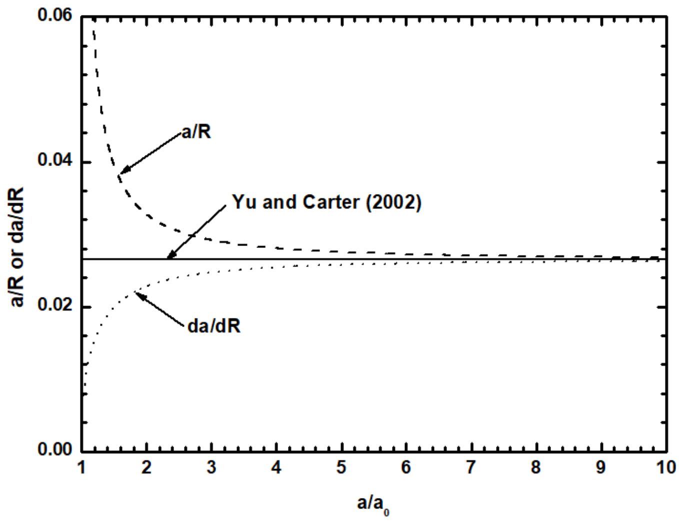

Figure 5 Varaition of a/R and da/dR for a cylindrical cavity

CONCLUSIONS

In this paper, the solutions for cavity expansion in cohesive frictional soil are compared. The soil is modeled as an elastic-perfectly plastic material according to the Mohr Coulomb failure criterion. Vesic (1972) solution was extended to account for a cavity expanding from a finite radius. The results show that the average volumetric strain has a significant influence in estimating the pressure expansion curve and limiting pressure for the generalised Vesic solution. The pressure expansion relationship for a cavity expansion from a finite radius was also obtained from a numerical procedure using Hill's incremental velocity approach. Further comparison shows that the proposed numerical procedure based on the Hill's incremental velocity method and Yu and Houlsby (1991) solution give practically identical results. The numerical results also confirm that a steady state configuration will be achieved when a cavity is expanded from an initial finite radius to infinity at the limit state, which gives equivalent ratio of plastic radius to the current cavity radius in a created cavity from zero initial radius. The incremental velocity approach and the suggested numerical procedure may be applied to cavity expansion in layered soils.

APPENDIX

Several functions and results presented in Vesic (1972), Carter et al. (1986) and Yu and Carter (2002) are reproduced herein for reference using the notation adopted in this paper.

$$I_r = \frac{E}{2(1+v)(c+p_0\tan\phi)} \qquad I_{rr} = \frac{I_r}{1+I_r\Delta} \qquad I_{rr} = \frac{I_r}{1+I_r\Delta\sec\phi}$$

$$G = \frac{E}{2(1+v)} \quad M = \frac{E}{1-v^2(2-k)} \quad Y = \frac{2c\cos\phi}{1-\sin\phi} \quad \alpha = \frac{1+\sin\phi}{1-\sin\phi} \quad \beta = \frac{1+\sin\psi}{1-\sin\psi} \quad \frac{p_1+c\cot\phi}{p_0+c\cot\phi} = \frac{1+k}{\alpha+k}\alpha$$

$$\chi = \frac{1}{M}\left[\beta - \frac{kv}{1-v(2-k)}\right] + \frac{1}{M\alpha}\left[k(1-2v)+2v-\frac{k\beta v}{1-v(2-k)}\right] \quad q = \frac{(1+k)\alpha[Y+(\alpha-1)p_0]}{(\alpha-1)(k+\alpha)} \quad s = -\frac{\chi q k(\alpha-1)}{\alpha\beta}$$

$$V_{r=a} = \frac{\mathrm{d}a}{\mathrm{d}R} = \exp\left[-\frac{\chi q}{\beta}\left(\frac{R}{a}\right)^{\frac{k(\alpha-1)}{\alpha}}\right]\left\{\sum_{n=0}^{\infty} A_n\left(\frac{R}{a}\right)^{\frac{k(\alpha-1)(1+n)}{\alpha}-1} + \left[\varepsilon_R(1+k)\exp\left(\frac{\chi q}{\beta}\right) - \sum_{n=0}^{\infty} A_n\right]\left(\frac{R}{a}\right)^{\frac{k}{\beta}}\right\} \tag{9}$$

in which A_n is defined by

$$A_n = \frac{1}{n!}\left(\frac{\chi q}{\beta}\right)^n \frac{\alpha\beta s}{k\alpha - k\beta(\alpha-1)(1+n)+\alpha\beta}$$

$$A_n = \frac{s}{n!}\left(\frac{\chi q}{\beta}\right)^n \ln\frac{a}{R} \qquad \text{if} \qquad \frac{k}{\beta} = \frac{k(\alpha-1)(1+n)}{\alpha} - 1 \tag{10}$$

and n is an integer ranging from 0 to infinity.
Note that Eq (10) was not provided by Yu and Carter (2002). It was derived by the authors.

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